Optimal insurance under smooth ambiguity

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Abstract

*We **proved the existence** of an optimal insurance contract under smooth ambiguity under a mild set of assumptions. We found that **under ambiguity neutrality, the shape of the optimal insurance contract is unaffected**. In the binary state case with risk-neutral insurer and constant loading, some **co-insurance might arise due to ambiguity aversion** and the failure to include the ambiguous states in the contract.*

1. Introduction

What is ambiguity?

- Incomplete information on the distribution of outcomes.
- Ellsberg (1961) and subsequent experimental evidence: subjects disproportionately prefer a risky (unambiguous) situation to an ambiguous one.

How to model ambiguity?

Multiple priors approach:

- Maximin model *à la* Gilboa & Schmeidler (1989): comparing acts according to minimum expected utility. Too pessimistic?
- Smooth ambiguity model *à la* Klibanoff et al. (2005): attitudes towards ambiguity described by a functional ϕ. Ambiguity-averse/neutral/seeking if ϕ is concave/linear/convex.
- Measure of ambiguity aversion analogous to Arrow-Pratt's of risk aversion.

Optimal insurance under risk

Raviv (1979) synthesized previous works of Borch (1960) and Arrow (1974). Key findings:

- Straight deductibles optimal under risk-neutral insurer and constant loading (linear cost of indemnity provision).
- Co-insurance arising due to either risk-aversion of the insurer or convex cost of indemnity provision.

Optimal insurance under risk and ambiguity

- Recent similar works of Gollier (2014), Alary et al. (2013): Disappearing deductibles optimal under risk-*cum*-ambiguity neutral insurer.
- Drawbacks: existence issue, ambiguity aversion on the part of the insurer.

2. Hypotheses and formulation

Assumptions

- **Prior distribution**. Prior belief in the probability of state i occurring is p_i, where $i \in \mathcal{I} = \{1, 2, \ldots, n\}$, $p_i \in (0,1)$ and $\sum_{i=1}^n p_i = 1$.
- **Conditional distributions**. The loss random variable $\tilde{x}$ has cdfs $F_i(\cdot) : I_x \to [0,1]$. All cdfs are C^2 on the common bounded support $I_x = [0, \bar{x}]$, and the pdfs $f_i(x) = \frac{\partial F_i(x)}{\partial x} > 0$ for each $i \in \mathcal{I}$.
- **MLR dominance**. For all $i, j \in \mathcal{I}$, the likelihood ratio $\ell_{ij} = \frac{f_i(x)}{f_j(x)}$ is nonincreasing in x (strictly decreasing on some positive-measured subset of I_x) whenever $i < j$. In other words $F_i(\cdot) \succcurlyeq_{MLR} F_j(\cdot)$ for all $i < j$.
- **Ambiguity aversion**. The welfare functional $\phi_J : \mathcal{V} \to \mathbb{R}$ is at least C^2, satisfying $\infty > \phi'_J > 0 \geq \phi''_J$, for each $J \in \{A, P\}$.
- **Risk aversion**. Attitudes towards risk are modeled by vNM utility functions. The policyholder is risk-averse; the insurer is risk-neutral.
- **Convex cost**. The cost of indemnity provision satisfies $\psi' > 0$ and $\psi'' \geq 0$.

Policyholder's problem

$$\max_{(I(\cdot),\pi)} \sum_{i=1}^n p_i \phi_A \left(\int_{I_x} u(W_A - \pi - x + I(x)) f_i(x) dx \right)$$

s.t. (P)

$$I(x) \in [0, x], \quad \forall x \in I_x,$$
$$\pi \in I_\pi = [\underline{\pi}, \overline{\pi}], \quad \underline{\pi} > 0,$$
$$\sum_{i=1}^n p_i \phi_P \left(\int_{I_x} (W_P + \pi - I(x) - \psi(I(x)) f_i(x) dx \right) \geq \bar{V},$$

where $\bar{V} = \phi_P(W_P)$ represents the insurer's outside options.

Proposition 1 (Existence)**.** *There exists an optimal insurance policy.*

3. Characterization of solution

Expected marginal welfares (EMWs)

- Bayesian posterior: $p_i(x) \equiv \frac{p_i f_i(x)}{\sum_i p_i f_i(x)}$.
- EMW of the policyholder: $A(x) \equiv \sum_i p_i(x) \phi'_A(y_i(\bar{x}))$.
- EMW of the insurer: $P(x) \equiv \sum_i p_i(x) \phi'_P(z_i(\bar{x}))$.
- Ratio of EMWs: $G(x) = \frac{A(x)}{P(x)}$, hence $\frac{G'(x)}{G(x)} = \frac{A'(x)}{A(x)} - \frac{P'(x)}{P(x)}$

Proposition 2 (Nondecreasing EMW)**.** *In the binary state case $n = 2$, **expected marginal welfares are nondecreasing in the loss**. Consequently, expected utility in the MLR dominant state is greater for both DMs in optimality. Furthermore, the ratio of EMWs is monotone in the loss.*

Corollary 1. *In the binary state case, the **ratio of EMWs is nondecreasing** if if insurer is ambiguity-neutral; it is nonincreasing if the policyholder is ambiguity-neutral.*

Intuitively, the more ambiguity-averse the policyholder and the less ambiguity-averse the insurer, the more likely is the ratio of EMWs to be nondecreasing in the loss.

Remark 1. *The **ratio of EMWs is constant under ambiguity neutrality**.*

The optimal insurance policy

Proposition 3. *In the binary state case with constant loading, risk-neutral insurer, if the policyholder is sufficiently more ambiguity-averse than the insurer (so that the EMWs ratio is nondecreasing), then optimal insurance involves **co-insurance below a threshold**. In particular, there exists thresholds x_1 and x_2 in $(0, \bar{x})$ such that the indemnity function has the form:*

$$\begin{cases} I(x) = 0 & x \in [0, x_1], \\ I(x) \in (0, x) & x \in (x_1, x_2), \\ I(x) = x & x \in [x_2, \bar{x}]. \end{cases} \tag{1}$$

Furthermore, for all the loss belonging to (x_1, x_2), the indemnity function satisfies the ODE:

$$I'(x) = \frac{r_u(W_A(x)) + \frac{G'(x)}{G(x)}}{r_u(W_A(x))}, \tag{2}$$

where $W_A(x) \equiv W_A - \pi - x + I(x)$ and $r_u(\cdot)$ is the Arrow-Pratt measure of absolute risk aversion.
The optimal premium is determined via the equality participation constraint.

4. Concluding remarks

Main findings

- There **exists an optimal insurance** contract under ambiguity with the standing assumptions (existence is robust to the case of risk-averse insurer).
- **Co-insurance** may arise even under constant loading and risk-neutral insurer when the policyholder is relatively more ambiguity-averse; type of contract resembles a stop-loss one in re-insurance.
- It can be shown that **if the state is contractible**, i.e., if the contract can be written as $(I_i(\cdot), \pi)_{i \in \mathcal{I}}$ then co-insurance disappears (Deductibles result holds state-wise).

Limitations and future research

- Generalization to more than two ambiguous states.
- Accounting for **moral hazard**.

References

Alary, D., Gollier, C. & Treich, N. (2013), 'The effect of ambiguity aversion on insurance and self-protection', *The Economic Journal* **123**(573), 1188–1202.

Arrow, K. J. (1974), 'Optimal insurance and generalized deductibles', *Scandinavian Actuarial Journal* **1974**(1), 1–42.

Borch, K. (1960), 'The safety loading of reinsurance premiums', *Scandinavian Actuarial Journal* **1960**(3-4), 163–184.

Ellsberg, D. (1961), 'Risk, ambiguity, and the savage axioms', *The quarterly journal of economics* pp. 643–669.

Gilboa, I. & Schmeidler, D. (1989), 'Maxmin expected utility with non-unique prior', *Journal of mathematical economics* **18**(2), 141–153.

Gollier, C. (2014), 'Optimal insurance design of ambiguous risks', *Economic Theory* **57**(3), 555–576.

Klibanoff, P., Marinacci, M. & Mukerji, S. (2005), 'A smooth model of decision making under ambiguity', *Econometrica* **73**(6), 1849–1892.

Raviv, A. (1979), 'The design of an optimal insurance policy', *The American Economic Review* **69**(1), 84–96.