Stochastic Deflator for an Economic Scenario Generator with Five Factors

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Abstract

We implement a stochastic deflator with four economic and financial risk factors: interest rates, stock prices, default intensities, and convenience yields.

We examine the deflator with different financial assets, such as stocks, zero-coupon bonds, vanilla options, and corporate coupon bonds.

Our numerical results show the reliability of the deflator approach in pricing financial derivatives.

Introduction

Due to the complicatedness of life insurance contracts and interactions among economic and financial risk factors, a reliable tool for asset/liability management (ALM) and calculations of reserves is demanded. In practice, "economic scenario generators" assist insurers in pricing insurance contracts and managing long-term risk.

Usually, economic scenarios are computed under a risk-neutral measure; the actualization process involving risk-free rate is quite simple. However, many "unusual" scenarios occur (e.g. 10-year rate 50%) under risk-neutral measure, which increases the difficulty to justify the calibration of "reaction functions" embedded in the ALM-projection model used to compute cash flows.

For example, the lapse rate is often a function of the difference between the revalorization rate of the contract and a reference rate; the parameters are calibrated observing "usual" values of economic parameters but may become difficult later to justify for atypical values of economic risk factors.

We could use a stochastic deflator to address this problem, using only scenarios under physical measure. Though the numerical calculations become tedious due to the complexity of the deflator, the benefit is that we could calculate the deflator separately and multiply the deflator with projected cash flows for pricing insurance contracts.

We adopt the deflator and include the processes of default and convenience yield to calculate prices for financial derivatives. We compare the values calculated from the deflator approach with the values suggested by analytical formulas in simple cases. Our goal is then to use this deflator to compute best estimates for a life insurance contract.

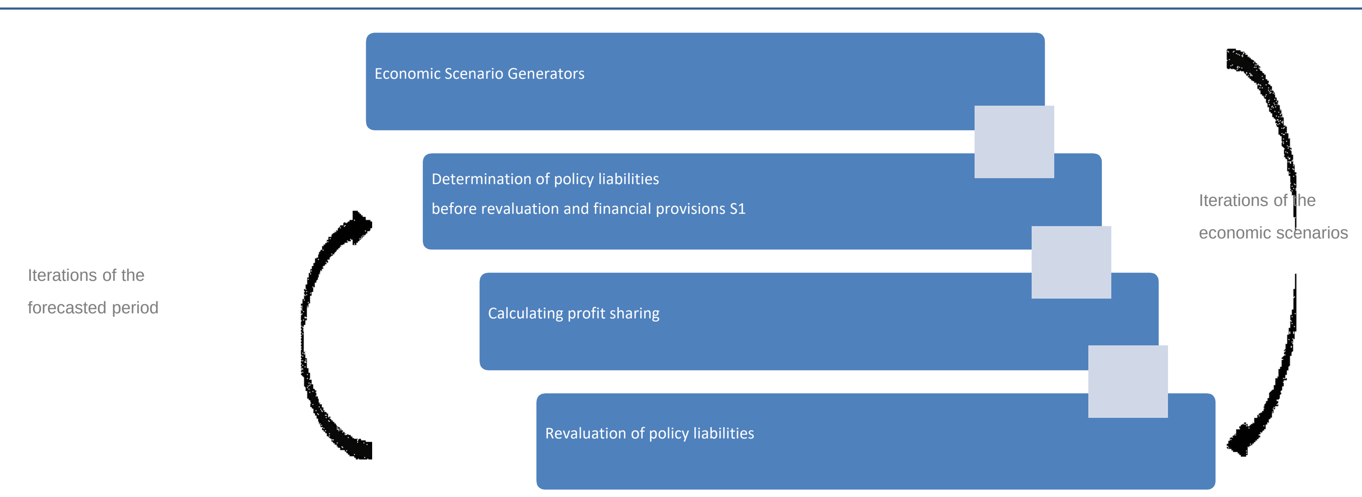


Figure 1. Calculating the best estimate reserve for a life insurance contract

Methodology

Step 1 Generate correlated Brownian motions

$$\mathbf{W}_{ESG}=\begin{bmatrix}W_r\\W_S\\W_\chi\\W_\gamma\end{bmatrix}=\begin{bmatrix}1&0&0&0\\\rho_{rS}&\sqrt{1-\rho_{rS}^2}&0&0\\\rho_{r\chi}&\rho'_{S\chi}&\rho'_{\chi\chi}&0\\\rho_{r\gamma}&\rho''_{S\gamma}&\rho''_{\chi\gamma}&\rho''_{\gamma\gamma}\end{bmatrix}\begin{bmatrix}W_0\\W_1\\W_2\\W_3\end{bmatrix}$$

$$\rho'_{S\chi}=\frac{\rho_{S\chi}-\rho_{rS}\rho_{r\chi}}{\sqrt{1-\rho_{rS}^2}},\ \rho'_{\chi\chi}=\sqrt{\frac{1-\rho_{rS}^2-\rho_{r\chi}^2-\rho_{S\chi}^2+2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}{1-\rho_{rS}^2}},$$

$$\rho''_{\chi\gamma}=\frac{\rho_{\chi\gamma}-\rho_{r\chi}\rho_{r\gamma}-\rho_{S\chi}\rho_{S\gamma}-\rho_{rS}^2\rho_{\chi\gamma}+\rho_{rS}\rho_{r\chi}\rho_{S\gamma}+\rho_{rS}\rho_{r\gamma}\rho_{S\chi}}{\sqrt{1+\rho_{rS}^4-2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi}-2\rho_{rS}^2+\rho_{rS}^2\rho_{r\chi}^2+\rho_{rS}^2\rho_{S\chi}^2-\rho_{r\chi}^2-\rho_{S\chi}^2+2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}},$$

$$\rho''_{\gamma\gamma}=\sqrt{1-\rho_{r\gamma}^2-\rho''^2_{S\gamma}-\rho''^2_{\chi\gamma}}.$$

Step 2 Derive general form of deflator with five factors

$$dD(t)=-D(t)r(t)dt-D(t)\theta(t)dW_r(t)+D(t)K_\Psi(t)dW_1(t)+D(t)K_\Gamma(t)dW_2(t)+D(t)K_\mathrm{I}(t)dW_3(t)$$

$$K_\Psi(t)=\frac{\Psi(t)}{D(t)}=\frac{r(t)+\theta(t)\sigma_S(t)\rho_{rS}-\mu_S(t)}{\sigma_S(t)\sqrt{1-\rho_{rS}^2}},\ K_\Gamma(t)=\frac{\Gamma(t)}{D(t)}=\frac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}}+\frac{r(t)\chi(t)-e+f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}}+\frac{\rho'_{S\chi}\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]}{\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}},$$

$$K_\mathrm{I}(t)=\frac{\mathrm{I}(t)}{D(t)}=\frac{\rho_{r\gamma}\theta(t)}{\rho''_{\gamma\gamma}}+\frac{r(t)\gamma(t)}{\eta\rho''_{\gamma\gamma}}-\frac{\rho''_{\chi\gamma}\rho_{r\chi}\theta(t)}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}}+\frac{\rho''_{\chi\gamma}\left[e-r(t)\chi(t)-f\chi(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_\chi\sqrt{\chi(t)}}+\frac{\left(\rho''_{S\gamma}\rho'_{\chi\chi}-\rho''_{\chi\gamma}\rho'_{S\chi}\right)\left[\mu_S(t)-r(t)-\rho_{rS}\theta(t)\sigma_S(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}}$$

Step 3 Implement time discretization

For a stochastic process $dX(t)=b_X(t,X(t))dt+\sigma_X(t,X(t))dW_X(t)$, *we partition the time* $[0,T]$ *into* N *segments with each length equalling* $(T-0)/N$, *then we have a time discretization* $\Pi_N=\Pi_N([0,T])$ *with* $0=t_0<t_1<\cdots<t_N=T$. *We approximate* $X(t)$ *by* Y_{i+1} *discretely,* $i=0,1,\ldots,N-1$.

Euler method

$$Y_{i+1}=Y_i+b_X(t_i,Y_i)(t_{i+1}-t_i)+\sigma_X(t_i,Y_i)(W_{i+1}-W_i)$$

Milstein method

$$Y_{i+1}=Y_i+b_X(t_i,Y_i)(t_{i+1}-t_i)+\sigma_X(t_i,Y_i)(W_{i+1}-W_i)+\frac{1}{2}\sigma_X(t_i,Y_i)\sigma_{X'}(t_i,Y_i)\left[(W_{i+1}-W_i)^2-(t_{i+1}-t_i)\right],$$

$$\sigma_{X'}=\frac{\partial\sigma_X(t,x)}{\partial x}$$

Simplified Second Milstein method

$$dX_{t,d\times1}=a_{d\times1}(t,X_t)dt+b_{d\times m}(t,X_t)dW_{t,m\times1}$$

$$Y_{n+1,i}=Y_{n,i}+a_i(n,Y_n)\Delta t+\sum_{k=1}^{m}b_{ik}(n,Y_n)\Delta W_{n,k}+\frac{1}{2}L^0a_i(n,Y_n)(\Delta t)^2$$
$$+\frac{1}{2}\sum_{k=1}^{m}\left[L^ka_i(n,Y_n)+L^0b_{ik}(n,Y_n)\right]\Delta W_{n,k}\Delta t+\frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}L^jb_{ik}(n,Y_n)\left(\Delta W_{n,j}\Delta W_{n,k}-V_{jk}\right)$$

$$L^0=\frac{\partial}{\partial t}+\sum_{i=1}^{d}a_i(t,X_t)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d}\Sigma_{t,ij}\frac{\partial^2}{\partial x_i\partial x_j};\ L^k=\sum_{i=1}^{d}b_{ik}(t,X_t)\frac{\partial}{\partial x_i},\ \forall k=1,\ldots,m$$

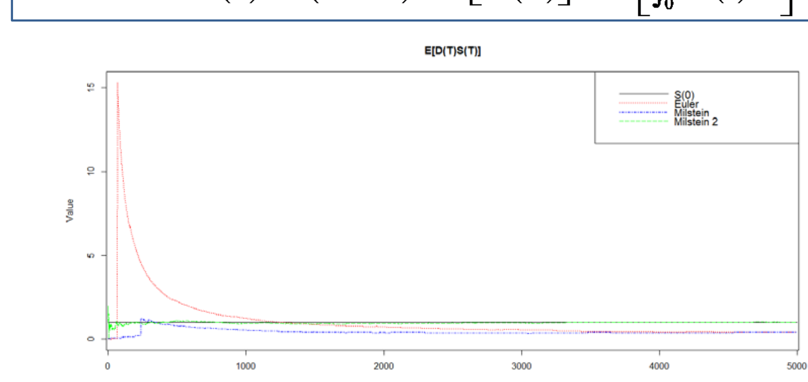
Numerical Results

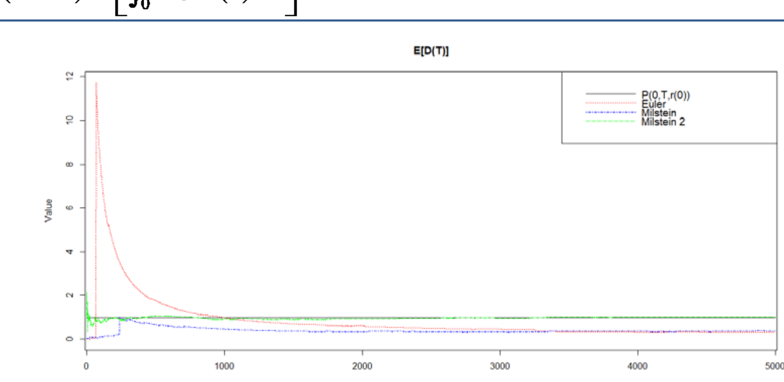
$$D(0)S(0)=S(0)=E^{\mathbf{Q}}\left[\delta(T)S(T)\right]=E\left[D(T)S(T)\right]=1$$

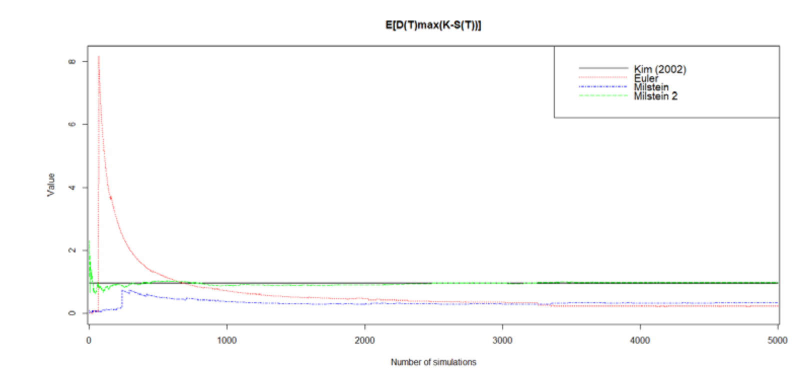
$$D(0)P(0,T,r(0))=P(0,T,r(0))=E^{\mathbf{Q}}\left[\delta(T)P(T,T,r(T))\right]=E\left[D(T)P(T,T,r(T))\right]=E\left[D(T)\right]$$

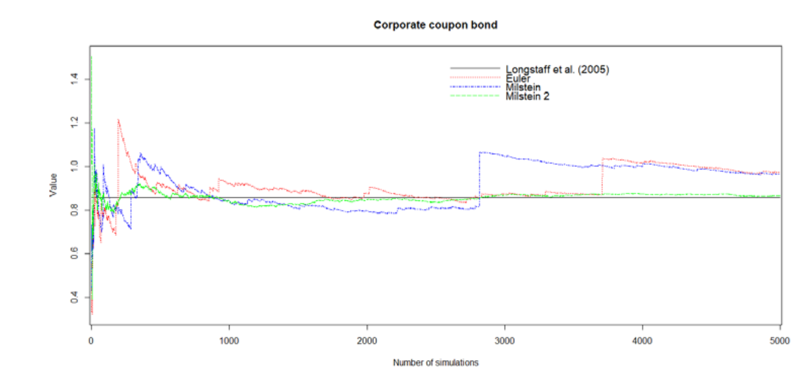
$$D(0)Put(0,S(0),T,K)=Put(0,S(0),T,K)=E^{\mathbf{Q}}\left[\delta(T)(K-S(T))^+\right]=E\left[D(T)(K-S(T))^+\right]$$

$$D(0)CB(c,\omega,T)=E\left[D(T)\right]+cE\left[\int_0^TD(t)dt\right]+(1-\omega)E\left[\int_0^T\chi_tD(t)dt\right]$$









Discussion

$$\begin{cases}dr(t)=\left[a_r-b_rr(t)\right]dt+\sigma_r\sqrt{r(t)}d\tilde{W}_r(t);\ a_r,b_r,\sigma_r>0\\d\theta(t)=\left[a_\theta-b_\theta\theta(t)\right]dt+\sigma_\theta\sqrt{\theta(t)}dW_\theta(t);a_\theta,b_\theta,\sigma_\theta>0\end{cases}$$

$$dr(t)=\left[a_r-b_rr(t)+\theta(t)\sigma_r\sqrt{r(t)}\right]dt+\sigma_r\sqrt{r(t)}dW_r(t)$$

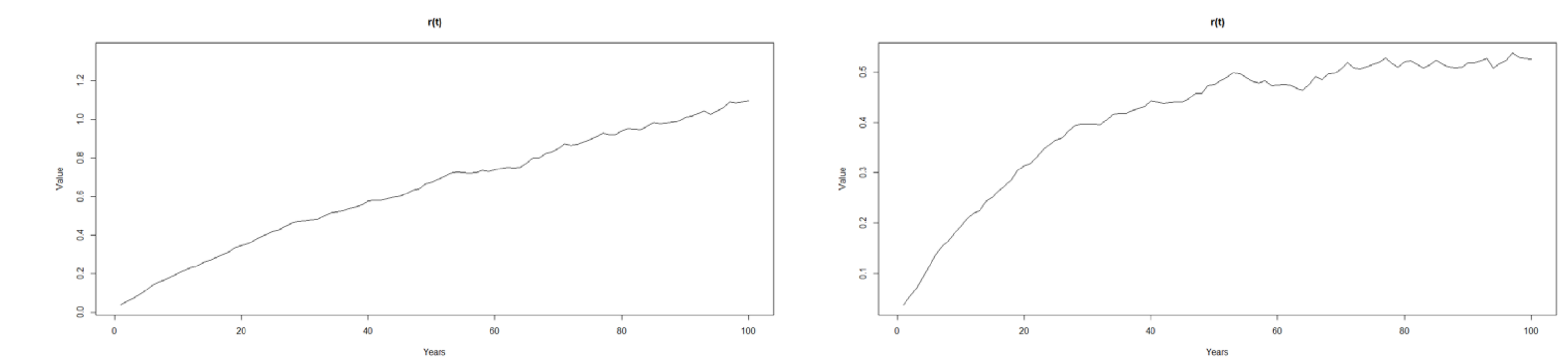


Figure 2. r(t) over long run with (right)/without (left) θ(t) being controlled

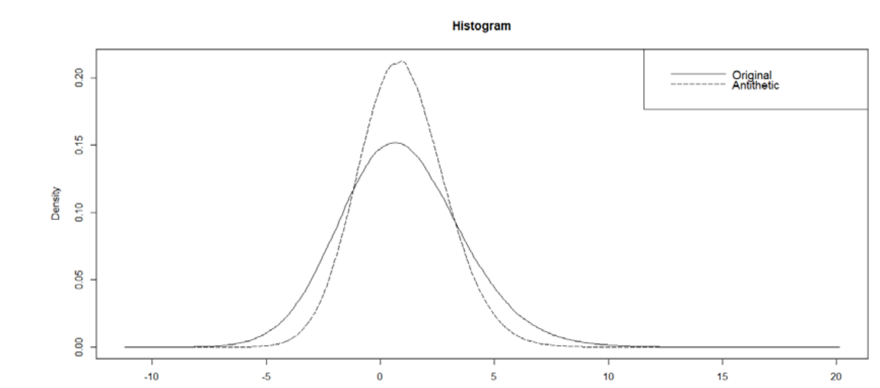


Figure 3. Histogram comparison of deflator with/without antithetic sampling

Conclusions

Our results indicate the reliability of the deflator for financial asset pricing, if the time discretization of the underlying stochastic processes is done carefully.

Except the benefit that we could compute best estimate value by simply averaging the multiplication of deflator and projected cash flows, the fact that we observe data only in physical world would provide the motivation for us to use deflator for the convenience to estimate parameters of "reaction functions" in an ALM projection model as in Chapter 4 of Laurent et al. (2016).

Further work would be to implement the deflator and compare the best estimate values of a life insurance contract under physical measure and risk-neutral measure.

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