Stochastic Deflator for an Economic Scenario Generator with Five Factors

Po–Keng Cheng Frédéric Planchet
Chaire DAMI, SAF, Université Lyon 1

Outline

- Intro (5 mins)
- Implementation (10 mins)
- Numerical results (5 mins)
- Q&A

Related literature review

- Arrow and Debreu (1954)
- Harrison and Kreps (1979)
- Dastarac and Sauveplane (2010); Caja and Planchet (2011)
- Bonnin et al. (2014); Borel-Mathurin et al. (2015); Vedani et al. (2017)

Motivations

- Many "unusual" scenarios occur (e.g. 10-year rate 50%) under risk-neutral measure, which increases the difficulty to justify the calibration of "reaction functions" embedded in the ALM-projection model used to compute cash flows
- Example

  The lapse rate is often a function of the difference between the revalorization rate of the contract and a reference rate; the parameters are calibrated observing "usual" values of economic parameters but may become difficult later to justify for atypical values of economic risk factors

Stochastic Deflator

- Benefit

We could compute best estimate value by simply averaging the multiplication of deflator and projected cash flows

$$E^{\mathbf{Q}}\left(\delta(t)X\right)=E^{\mathbf{P}}\left[D(t)X\right]$$

$\delta(t)$: discount factor

$D(t)$: stochastic deflator

X: nonnegative random variable

Summary

- In this paper, we implement a stochastic deflator with four economic and financial risk factors: interest rates, stock prices, default intensities, and convenience yields. We examine the deflator with different financial assets, such as stocks, zero-coupon bonds, vanilla options, and corporate coupon bonds.
- Challenges remain for finding better algorithm to approximate the process of deflator.

Brownian Motions

$$\mathbf{W}_{ESG} = \begin{bmatrix} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_{rS} & \sqrt{1-\rho_{rS}^2} & 0 & 0 \\ \rho_{r\chi} & \rho'_{S\chi} & \rho'_{\chi\chi} & 0 \\ \rho_{r\gamma} & \rho''_{S\gamma} & \rho''_{\chi\gamma} & \rho''_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

$$\rho'_{S\chi} = \frac{\rho_{S\chi} - \rho_{rS}\rho_{r\chi}}{\sqrt{1-\rho_{rS}^2}}, \ \rho'_{\chi\chi} = \sqrt{\frac{1-\rho_{rS}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}{1-\rho_{rS}^2}},$$

$$\rho''_{\chi\gamma} = \frac{\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma} - \rho_{S\chi}\rho_{S\gamma} - \rho_{rS}^2\rho_{\chi\gamma} + \rho_{rS}\rho_{r\chi}\rho_{S\gamma} + \rho_{rS}\rho_{r\gamma}\rho_{S\chi}}{\sqrt{1+\rho_{rS}^4 - 2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{r\chi}^2 + \rho_{rS}^2\rho_{S\chi}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}},$$

$$\rho''_{\gamma\gamma} = \sqrt{1-\rho_{r\gamma}^2 - \rho''^2_{S\gamma} - \rho''^2_{\chi\gamma}}.$$

SDEs

$$dr(t) = \alpha(t, r(t))dt + \beta(t, r(t))dW_r(t)$$

$$dB(t) = B(t)r(t)dt$$

$$dP(t, T, r(t)) = P(t, T, r(t))\left[r(t) + \tilde{\sigma}(t, r(t))\theta(t)\right]dt$$

$$+ P(t, T, r(t))\tilde{\sigma}(t, r(t))dW_r(t), \; \tilde{\sigma}(t, r(t)) = \frac{P_r \beta(t, r(t))}{P(t, r(t))}$$

$r(t)$: interest rate; $B(t)$: short-term saving

$P(t, T, r(t))$: zero coupon bond of no risk with maturity T

$\theta(t)$: market price of risk

SDEs

$$dS(t) = S(t)\mu_S(t)dt + S(t)\sigma_S(t)dW_S(t)$$

$$d\chi(t) = \left[e - f\chi(t)\right]dt + \sigma_\chi\sqrt{\chi(t)}dW_\chi(t)$$

$$d\gamma(t) = \eta dW_\gamma(t)$$

$S(t)$: stock price; $\chi(t)$: default density

$\gamma(t)$: convenience yield

SDE of Stochastic Deflator

$$dD(t)=\Omega\big(D(t),t,r(t)\big)dt+\Phi\big(D(t),t,r(t)\big)dW_r(t)+\Psi\big(D(t),t,r(t)\big)dW_1(t)$$
$$+\Gamma\big(D(t),t,r(t)\big)dW_2(t)+\mathrm{I}\big(D(t),t,r(t)\big)dW_3(t)$$

$$\Omega\big(D(t),t,r(t)\big)=-D(t)r(t),\ \Phi\big(D(t),t,r(t)\big)=-D(t)\theta(t),$$

$$\Psi\big(D(t),t,r(t)\big)=\frac{D(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}-\mu_S(t)\right]}{\sigma_S(t)\sqrt{1-\rho_{rS}^2}},$$

$$\Gamma\big(D(t),t,r(t)\big)=D(t)\left[\begin{array}{l}\dfrac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}}+\dfrac{r(t)\chi(t)-e+f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}}\\ +\dfrac{\rho'_{S\chi}\left(\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right)}{\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}}\end{array}\right],$$

$$\mathrm{I}\big(D(t),t,r(t)\big)=D(t)\left\{\begin{array}{l}\dfrac{\rho_{r\gamma}\theta(t)}{\rho''_{\gamma\gamma}}+\dfrac{r(t)\gamma(t)}{\eta\rho''_{\gamma\gamma}}-\dfrac{\rho''_{\chi\gamma}\rho_{r\chi}\theta(t)}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}}+\dfrac{\rho''_{\chi\gamma}\left[e-r(t)\chi(t)-f\chi(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_\chi\sqrt{\chi(t)}}\\ +\dfrac{\left(\rho''_{S\gamma}\rho'_{\chi\chi}-\rho''_{\chi\gamma}\rho'_{S\chi}\right)\left[\mu_S(t)-r(t)-\rho_{rS}\theta(t)\sigma_S(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_S(t)\sqrt{\left(1-\rho_{rS}^2\right)}}\end{array}\right\}$$

Time Discretization

- Euler method, $i = 0,1,\dots,N-1$

$$Y_{i+1} = Y_i + b_X\left(t_i, Y_i\right)\left(t_{i+1} - t_i\right) + \sigma_X\left(t_i, Y_i\right)\left(W_{i+1} - W_i\right)$$

- Milstein method, $i = 0,1,\dots,N-1$

$$Y_{i+1} = Y_i + b_X\left(t_i, Y_i\right)\left(t_{i+1} - t_i\right) + \sigma_X\left(t_i, Y_i\right)\left(W_{i+1} - W_i\right)$$

$$+\frac{1}{2}\sigma_X\left(t_i, Y_i\right)\sigma_x\left(t_i, Y_i\right)\left[\left(W_{i+1} - W_i\right)^2 - \left(t_{i+1} - t_i\right)\right]$$

Time Discretization

- Second Milstein method, $i = 0,1,\ldots, N-1$

$$dX_{t,d\times 1} = a\left(t, X_t\right)_{d\times 1} dt + b\left(t, X_t\right)_{d\times m} dW_{t,m\times 1}$$

$$df\left(t, X_t\right) = \left[\frac{\partial f\left(t, X_t\right)}{\partial t} + \sum_{i=1}^{d} \frac{\partial f\left(t, X_t\right)}{\partial x_i} a_i\left(t, X_t\right) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f\left(t, X_t\right)}{\partial x_i \partial x_j} \Sigma_{t,ij}\right] dt$$

$$+ \sum_{i=1}^{d}\sum_{k=1}^{m} b_{ik}\left(t, X_t\right) \frac{\partial f\left(t, X_t\right)}{\partial x_i} dW_{t,k},\ \Sigma_t = b\left(t, X_t\right) b^T\left(t, X_t\right)$$

Time Discretization

- Second Milstein method (continue)

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{d} a_i\left(t, X_t\right)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \Sigma_{t,ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

$$L^k = \sum_{i=1}^{d} b_{ik}\left(t, X_t\right)\frac{\partial}{\partial x_i}, \ \forall k = 1,\ldots,m$$

$$df\left(t, X_t\right) = L^0 f\left(t, X_t\right)dt + \sum_{k=1}^{m} L^k f\left(t, X_t\right)dW_{t,k}$$

Time Discretization

- Second Milstein method (continue)

$$Y_{n+1,i} = Y_{n,i} + a_i(n, Y_n)\Delta t + \sum_{k=1}^{m} b_{ik}(n, Y_n)\Delta W_{n,k} + \frac{1}{2}L^0 a_i(n, Y_n)(\Delta t)^2$$

$$+ \frac{1}{2}\sum_{k=1}^{m}\left[L^k a_i(n, Y_n) + L^0 b_{ik}(n, Y_n)\right]\Delta W_{n,k}\Delta t$$

$$+ \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} L^j b_{ik}(n, Y_n)\left(\Delta W_{n,j}\Delta W_{n,k} - V_{jk}\right)$$

V_{jk} independent with $\mathbf{Pr}\left(V_{jk} = \Delta t\right) = \mathbf{Pr}\left(V_{jk} = -\Delta t\right) = \frac{1}{2}$ for $j < k$,

$V_{kj} = -V_{jk}$ for $j > k$, $V_{jk} = \Delta t$ for $j = k$

Numerical results

$$dr(t) = \left[a_r - b_r r(t)\right]dt + \sigma_r \sqrt{r(t)}\, d\tilde{W}_r(t);\ a_r, b_r, \sigma_r > 0$$

$$d\tilde{W}_r(t) = \theta(t)dt + dW_r(t)$$

Then,

$$dr(t) = \left[a_r - b_r r(t) + \theta(t)\sigma_r \sqrt{r(t)}\right]dt + \sigma_r \sqrt{r(t)}\, dW_r(t)$$

$$d\theta(t) = \left[a_\theta - b_\theta \theta(t)\right]dt + \sigma_\theta \sqrt{\theta(t)}\, dW_\theta(t); a_\theta, b_\theta, \sigma_\theta > 0$$

Numerical results

$$D(0)S(0)=S(0)=E^{\mathbf{Q}}\left[\delta(T)S(T)\right]=E^{\mathbf{P}}\left[D(T)S(T)\right]=1$$

$$D(0)P\left(0,T,r(0)\right)=P\left(0,T,r(0)\right)$$
$$=E^{\mathbf{Q}}\left[\delta(T)P\left(T,T,r(T)\right)\right]=E^{\mathbf{P}}\left[D(T)P\left(T,T,r(T)\right)\right]=E^{\mathbf{P}}\left[D(T)\right]$$

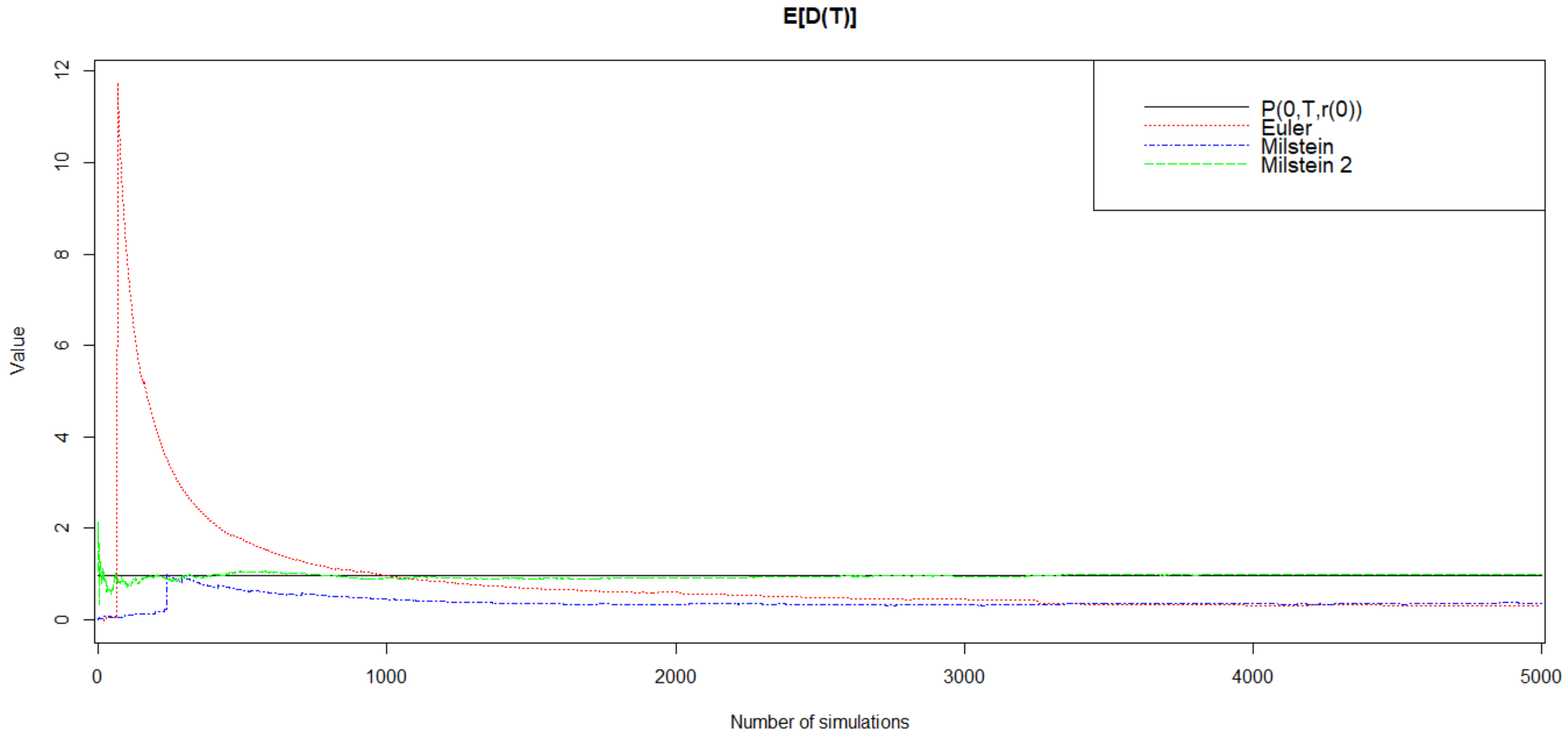
$$D(0)Put\left(r(0),S(0)\right)=Put\left(r(0),S(0)\right)$$
$$=E^{\mathbf{Q}}\left[\delta(T)\left(K-S(T)\right)^{+}\right]=E^{\mathbf{P}}\left[D(T)\left(K-S(T)\right)^{+}\right]$$

Numerical results

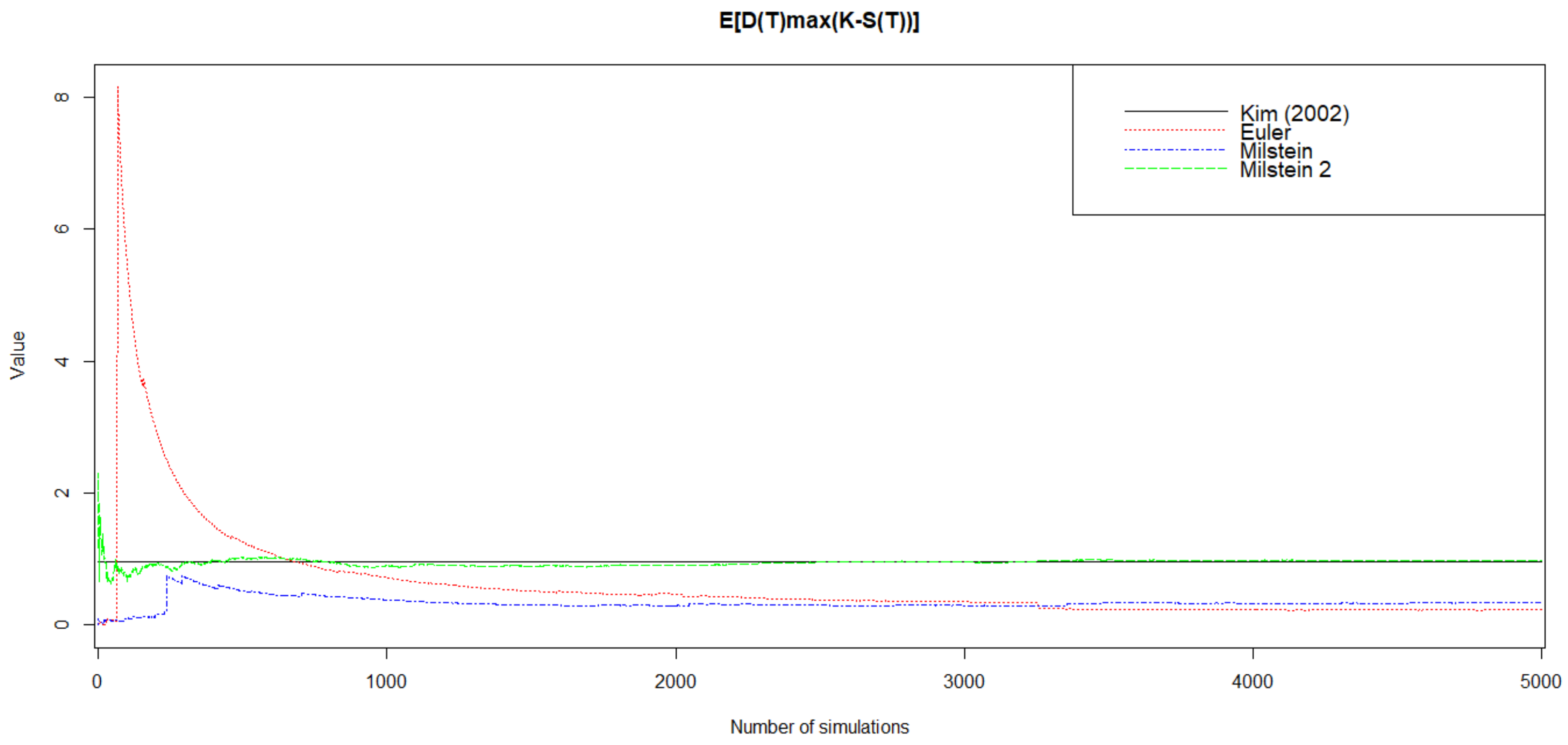
Tab. 2. Zero coupon bond of no risk with maturity T

# of Simulations	$E[D(T)]$	$E[D(T)P(T,T,r(T))]$	$P(0,T,r(0))$
Euler method			
2500	0.48812434395029	0.488066560145415	
5000	0.308038513966897	0.308003081851407	
10000	0.325024763106268	0.324990036772998	
100000	0.625985956594468	0.625922227828396	0.970957220487724
250000	0.771419872927604	0.771346645830578	
500000	0.726182177029966	0.72611342647583	
1000000	1.18961111433854	1.18949806538064	
Milstein method			
2500	0.332402071677238	0.332366375930865	
5000	0.366969978004539	0.366931013968048	
10000	0.360964352720743	0.360926322954742	
100000	0.654420842856107	0.654354469594408	0.970957220487724
250000	0.746857198318386	0.746783090935181	
500000	0.824192927498608	0.82411348941373	
1000000	1.15968204198937	1.15956727600934	
Second Milstein method			
2500	0.952656631120277	0.952657151589584	
5000	0.986807314190818	0.986807555270611	
10000	1.00104803675261	1.00104821737114	
100000	0.97943249260489	0.979432483928667	0.970957220487724
250000	0.974542088257683	0.974542020575479	
500000	0.975959099152459	0.975959075808478	
1000000	0.973919234689554	0.97391925959157	

Numerical results



Numerical results



Numerical results

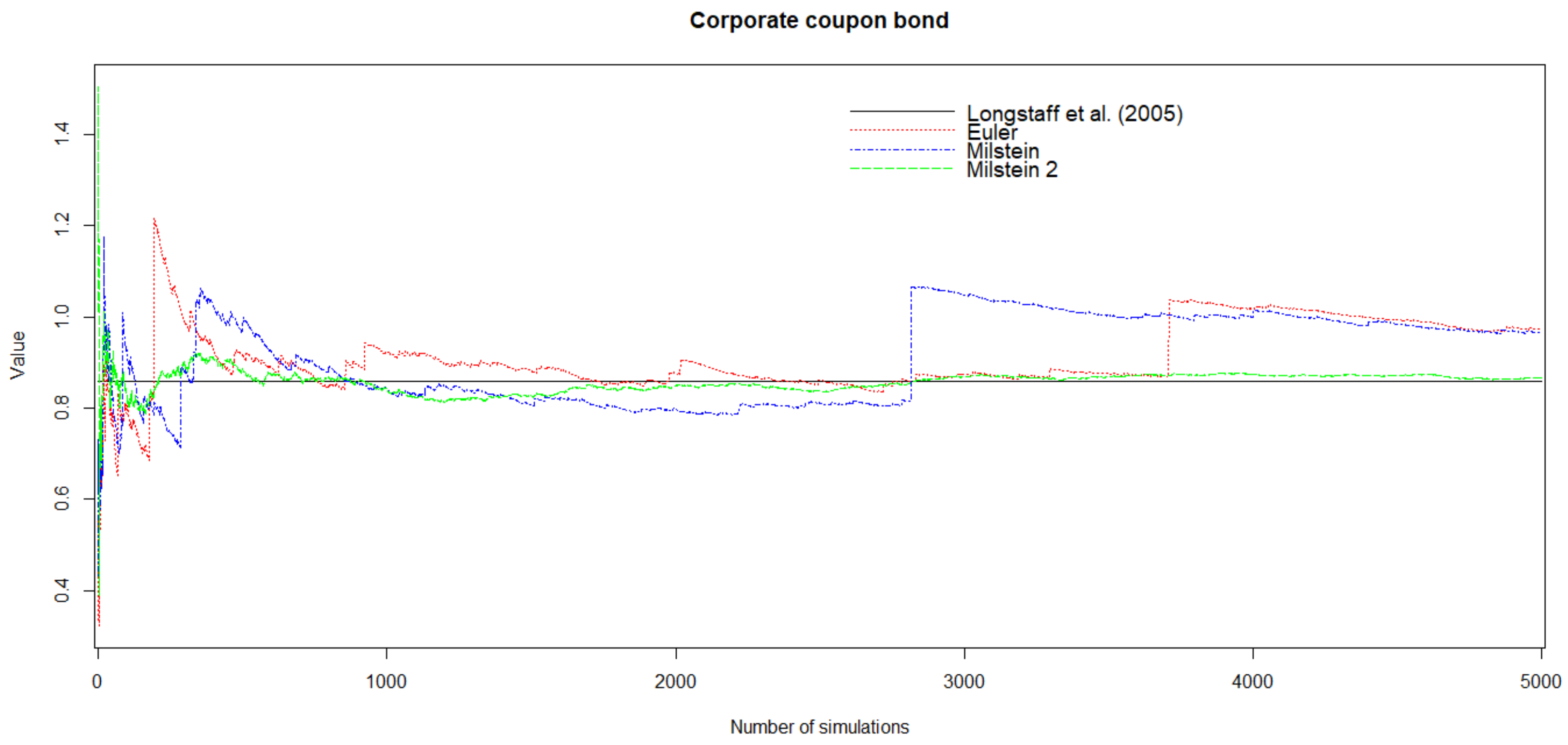
Longstaff et al. (2005)

$$\rho_{r\chi}=0,\ \rho_{r\gamma}=0,\ \rho_{S\chi}=0,\ \rho_{S\gamma}=0,\ \rho_{\chi\gamma}=0$$

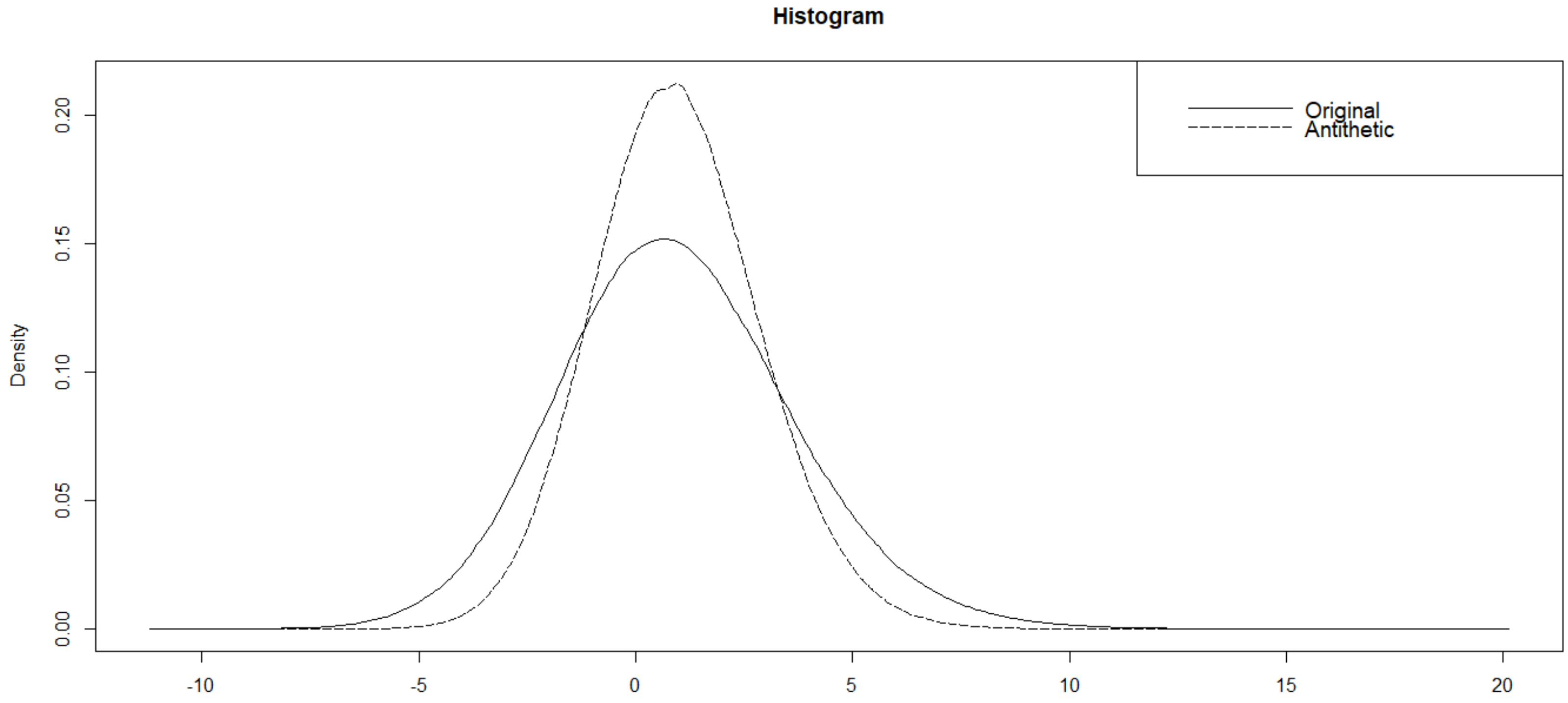
$$dB(t)=B(t)\left[r(t)+\chi(t)+\gamma(t)\right]dt$$

$$D(0)CB(c,\omega,T)=E^{\mathbf{P}}\left[D(T)\right]+cE^{\mathbf{P}}\left[\int_0^T D(t)\,dt\right]+(1-\omega)E^{\mathbf{P}}\left[\int_0^T \chi_t D(t)\,dt\right]$$

Numerical results



Antithetic Sampling



Confidence Interval

$$CI_{95\%} = E^{\mathbf{P}}\left[D(t)\right] + \frac{Var\left(E^{\mathbf{P}}\left[D(t)\right]\right)}{2} \pm t_{d.f.=n}\sqrt{\frac{Var\left(E^{\mathbf{P}}\left[D(t)\right]\right)}{n} + \frac{\left[Var\left(E^{\mathbf{P}}\left[D(t)\right]\right)\right]^2}{2(n-1)}}$$

Second Milstein method with antithetic sampling and sample size equalling 2500

$CI_{95\%}$ of $E^{\mathbf{P}}\left[D(T)\right] = [0.95192, 0.95559]$

Q&A

Thank you!