Stochastic State Space Models for Mortality

Dr. Gareth W. Peters

Chair Prof. of Statistics for Risk and Insurance
Department of Actuarial Mathematics and Statistics

Heriot-Watt University, Edinburgh, UK

D. Toczydlowska, G.W. Peters, M.C. Fung, P.V. Shevchenko.

"Stochastic Period and Cohort Effect State-Space Mortality Models Incorporating Demographic Factors via Probabilistic Robust Principle Components"

Risks (to appear 2017)

September 14, 2017

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

Mortality Modelling Context

The modelling and management of systematic mortality risk are two of the main concerns of large life insurers and pension plans:

Modelling:

- What is the best way to *forecast future mortality rates* and to model the *uncertainty surrounding these forecasts*?
- How do we value risky future cashflows that depend on future mortality rates?

Management:

- How can this risk be actively managed and reduced as part of an overall strategy of efficient risk management?
- What hedging instruments are easier to price than others?

Mortality Modelling Context

Stylized Facts of Mortality Data: Enhancing mortality models requires an understanding of common features of mortality behaviour (see discussion in [Cairns, Blake and Dowd, 2008])

- Mortality rates have fallen dramatically at all ages.
- Rate of decrease in mortality has varied over time and by age group.
  - *For example, for English and Welsh males, the age 25 rate improved dramatically before 1960 and then levelled off; conversely at age 65 the opposite was true*
- Absolute decreases have varied by age group.
  - *For example, for English and Welsh males, the age 45 improvements have been much higher than the age 85 improvements.*
- Aggregate mortality rates have significant volatility year on year.

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

Stochastic Mortality Models

The uncertainty in future death rates can be divided into two components:

- **Unsystematic mortality risk**. Even if the true mortality rate is known, the number of deaths, $D(t, x)$, will be random.
  - larger population $\Rightarrow$ smaller unsystematic mortality risk (due to pooling of offsetting risks - diversification).
- **Systematic mortality risk**. This is the undiversifiable component of mortality risk that affects all individuals in the same way.
  - **Forecasts of mortality rates in future years are uncertain.**

Background on Stochastic Mortality Modelling

- Pricing of retirement income products depends crucially on the accuracy of the predicted death or survival probabilities.

- *It has been widely documented that* **survival probability is consistently underestimated** *especially in the last few decades ([IMF, 2012]).*

- To capture the stochastic nature of mortality trends, [Lee and Carter, 92] proposed a stochastic mortality model to forecast the trend of age-specific mortality rates.

- **Since the introduction of the Lee-Carter model, a range of stochastic mortality models have been proposed in the literature.**

Stochastic Mortality Models

Single age group models:

- Model the individual age group mortality evolution either: force of mortality [1] or annual death counts.
- Typically such models include:
  - *temporal smoothing splines*;
  - *demographic factors*;
  - can be *count processes or functional regressions* (or both);
  - *ARIMA* type structures.

Term structure of mortality (multiple age group) models:

- Typically model the log mortality rate across the term structure of mortality.
- Typically such models include:
  - *temporal smoothing splines*;
  - *period effects*; and
  - *cohort effects*.

[1] force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. It is identical in concept to failure rate, also called hazard function, in reliability theory.

Stochastic Mortality Models: Regression Formulations

Generalized Linear Model Type: have been widely adopted in mortality modelling (eg.[Forfar, 1988], [Renshaw, 2003,2000,1991], [Currie,2016]).

Modelling target was either:

- the probability of death q_x, based on initial exposures; or
- the force of morality μ_x, based on central exposures.

When targeting q_x it was common to use a **binomial observation distribution**

When targeting the force of mortality it was common practice to use a **Poisson observation distribution**

Stochastic Mortality Models: Time-series Regression

Stochastic Period Effect Models: influential stochastic factor model for mortality modelling given by [Lee and Carter, 1992]

- Dynamics of the log crude death rates, $y_{x,t} = \ln \hat{m}_{x,t}$, follow:

$$y_{x,t} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}, \quad \varepsilon_{x,t} \overset{iid}{\sim} \mathrm{N}(0, \sigma_\varepsilon^2),$$

  - $\boldsymbol{\alpha} = \alpha_{x_1:x_p} := [\alpha_{x_1}, \ldots, \alpha_{x_p}]$ represents the *age-profile of the log death rates*
  - $\boldsymbol{\beta} = \beta_{x_1:x_p}$ measures the *sensitivity of of death rates for different age group* to a change of period effect κ_t.

- The **period effect**, κ_t, *for forecasting*, is typically set as

$$\kappa_t = \kappa_{t-1} + \theta + \omega_t, \quad \omega_t \overset{iid}{\sim} \mathrm{N}(0, \sigma_\omega^2),$$

where $\varepsilon_{x,t}$ and ω_t are independent.

Stochastic Mortality Models: Time-series Regression

Extensions to the LC model:

Model	Dynamics
[Lee and Carter,92]	$\ln(\hat{m}_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$
[Renshaw et al, 03]	$\ln(\hat{m}_{x,t}) = \alpha_x + \sum_{i=1}^{k} \beta_x^{(i)} \kappa_t^{(i)} + \varepsilon_{x,t}$
[Renshaw et al, 06]	$\ln(\hat{m}_{x,t}) = \alpha_x + \beta_x^{(1)} \kappa_t + \beta_x^{(2)} \zeta_{t-x} + \varepsilon_{x,t}$
[Currie,06]	$\ln(\hat{m}_{x,t}) = \alpha_x + \kappa_t + \zeta_{t-x} + \varepsilon_{x,t}$
[Cairns et al.,06]	$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$
[Cairns et al., 09]	$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \zeta_{t-x}$
[Plat, 09]	$\ln(\hat{m}_{x,t}) = \alpha_x + \kappa_t^{(1)} + \kappa_t^{(2)}(\bar{x} - x) + \kappa_t^{(3)}(\bar{x} - x)^+ + \zeta_{t-x} + \varepsilon_{x,t}$

Recently, [Fung et al, 16] and [Fung et al, 17] have proposed new extensions based on stochastic volatility structures in the latent processes as well as non-homoscedasticity in the mortality term structures, long-memory persistance and demographic factor model distributed lags.

Background on Stochastic Mortality Modelling

Frequentist Models with demographic and economic data.

- In [Hanewald,2011] and [Niu,2014] investigate links between *economic growth and morality trends* via regression model:
  - **Single Age Group Period effect Lee-Carter model + covariate (GDP).**
- [Hanewald,2011] included *cause-of-death* categorical variables.
- [Murray and Lopez, 1997] developed a multi factor linear regression model: the *logarithm of the rate of mortality per age group, sex and clustered cause of death* is regressed against the *socio-economic, educational, technological and cause-of-death* related regressors.
- [Hyndman and Yasmeen, 2012] and [Erbas, 2010] considered dimension reduction based feature extraction methods for regressors: *functional PCA covariates from mortality curves*

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

State Space Based Stochastic Mortality Models

A key advantage of state space modelling is that the typical two-stage estimation and forecasting procedure under the SVD or Poisson regression maximum likelihood approaches can be combined in a single setting. This has the following advantages:

- more numerically and statistically robust than standard two stage regression modelling;
- can remove awkward identification specifications;
- is computationally more efficient; and
- can produce more accurate in-sample and out-of-sample forecasts;
- can be optimal from an efficiency and unbiased estimation perspective;
- easily adapted to Bayesian inference!

State Space Based Stochastic Mortality Models

State-Space Formulation: Observation Process

Let $y_x = \ln \widetilde{m}_{x,t}$, in matrix notation we have (recall that $\gamma_t^x := \gamma_{t-x}$)

$$\begin{pmatrix} y_{x_1,t} \\ y_{x_2,t} \\ \vdots \\ y_{x_p,t} \end{pmatrix} = \begin{pmatrix} \alpha_{x_1} \\ \alpha_{x_2} \\ \vdots \\ \alpha_{x_p} \end{pmatrix} + \begin{pmatrix} \beta_{x_1} & \beta_{x_1}^{\gamma} & 0 & \cdots & 0 \\ \beta_{x_2} & 0 & \beta_{x_2}^{\gamma} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{x_p} & 0 & 0 & \cdots & \beta_{x_p}^{\gamma} \end{pmatrix} \begin{pmatrix} \kappa_t \\ \gamma_t^{x_1} \\ \gamma_t^{x_2} \\ \vdots \\ \gamma_t^{x_p} \end{pmatrix} + \begin{pmatrix} \varepsilon_{x_1,t} \\ \varepsilon_{x_2,t} \\ \vdots \\ \varepsilon_{x_p,t} \end{pmatrix}.$$

It is clear that, we have for $i \in \{1, \ldots, p\}$:

$$y_{x_i,t} = \alpha_{x_i} + \beta_{x_i}\kappa_t + \beta_{x_i}^{\gamma}\gamma_t^{x_i} + \varepsilon_{x_i,t}$$

Here $(\kappa_t, \gamma_t^{x_1}, \ldots, \gamma_t^{x_p})^\top$ is the $p+1$ dimensional latent state vector.

State Space Based Stochastic Mortality Models

State-Space Formulation: State Process

An extended latent cohort dynamic for $\gamma_t^{x_1}$ is obtained by specifying the second row of the $p+1$ by $p+1$ matrix.

For example, one can consider generally the state equation as

$$\begin{pmatrix} \kappa_t \\ \gamma_t^{x_1} \\ \gamma_t^{x_2} \\ \vdots \\ \gamma_t^{x_{p-1}} \\ \gamma_t^{x_p} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{p-1} & \lambda_p \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \kappa_{t-1} \\ \gamma_{t-1}^{x_1} \\ \gamma_{t-1}^{x_2} \\ \vdots \\ \gamma_{t-1}^{x_{p-1}} \\ \gamma_{t-1}^{x_p} \end{pmatrix} + \begin{pmatrix} \theta \\ \eta \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \omega_t^{\kappa} \\ \omega_t^{\gamma} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

where

$$\gamma_t^{x_1} = \lambda_1 \gamma_{t-1}^{x_1} + \lambda_2 \gamma_{t-1}^{x_2} + \cdots + \lambda_{p-1} \gamma_{t-1}^{x_{p-1}} + \lambda_p \gamma_{t-1}^{x_p} + \eta + \omega_t^{\gamma}$$

which is an ARIMA(p,0,0) process since $\gamma_{t-1}^{x_i} = \gamma_{t-i}^{x_1}$, $i = 2, \ldots, p$.

State Space Based Stochastic Mortality Models

Cohort effects: state-space formulation

We can express the matrix form succinctly as

$$\boldsymbol{y}_t = \boldsymbol{\alpha} + B\boldsymbol{\varphi}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \overset{iid}{\sim} \mathrm{N}(0, \sigma_\varepsilon^2 1_p),$$

$$\boldsymbol{\varphi}_t = \Lambda \boldsymbol{\varphi}_{t-1} + \boldsymbol{\Theta} + \boldsymbol{\omega}_t, \quad \boldsymbol{\omega}_t \overset{iid}{\sim} \mathrm{N}(0, \Upsilon),$$

where

$$B = \begin{pmatrix} \beta_{x_1} & \beta_{x_1}^{\gamma} & 0 & \cdots & 0 \\ \beta_{x_2} & 0 & \beta_{x_2}^{\gamma} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{x_p} & 0 & 0 & \cdots & \beta_{x_p}^{\gamma} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta \\ \eta \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

and $\boldsymbol{\varphi}_t = (\kappa_t, \gamma_t^{x_1}, \ldots, \gamma_t^{x_p})^\top$, 1_p the p-dimensional identity matrix and Υ is a $p+1$ by $p+1$ diagonal matrix with diagonal $(\sigma_\kappa^2, \sigma_\gamma^2, 0, \ldots, 0)$.

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

State-Space Hybrid Factor Models for Mortality

- Extending stochastic mortality models with observable exogenous features/covariates from demographic data.

- This offers two advantages to standard Lee-Carter models:
  - *firstly they may improve predictive power of the models*
  - *secondly they may improve interpretation of the dynamic of the "term-structure" of age specific mortality rates.*

We address four new and important aspects in practice previously ignored:

1. **missing data** in time-series and panel (matrix) structured real demographic data;
2. **noisy observations and outliers** (in real data);
3. **parsimonious model** creation via dimension reduction; and
4. **optimal estimation** via **computational efficient** state-space filtering methods.

State-Space Hybrid Factor Models for Mortality

"Hybrid": a mix of observable stochastic features and latents stochastic factors

- Two fundamental approaches to develop Hybrid Factor Models:
  1. time varying factor with static loading coefficient (*classical distributed lag regressions such as ARDL models*);
  2. static factor with time varying stochastic loading coefficients. (*state space models e.g. dynamic Nelson-Siegel yield curves*).
- Approach 2 is more appropriate for data which is **high dimensional** in nature, **time series / panel structured** but represented by relatively "short time series" lengths.
  - *This type of data is particularly prevalent in demographic studies!*
- The main concept here is that the **feature extraction** is performed over the entire available time series of observable demographic data.

State-Space Hybrid Factor Models for Mortality

- There are numerous ways to achieve this in a state-space model:
  - the *factor may influence all age groups equally* by entering the factor into the **state equation**; or
  - the *factor may influence each age specific mortality rate differently* by adding it in the **observation equation**.
- Denote generically $\mathbf{F}_t$ as the $p \times k$ factors matrix where p may represent number of age groups and k may represent number of age specific factors.
- We then specify an additional latent pk dimensional vector ϱ_t for the factor loading for year t.
  - ϱ_t is a dynamic regression parameter for factors matrix $\mathbf{F}_t$ which specifies the impact of $x_i \in \{x_1, \ldots, x_p\}$ age group and $m \in \{1, \ldots, k\}$ component corresponding to $[\mathbf{F}_t]_{i,m}$ by $\varrho_t^{i,m}$ element.

State-Space Hybrid Factor Models for Mortality

Consider the State-Space Hybrid Stochastic Period-Cohort-Demographic Model

The general notation of the model is as follows

$$\boldsymbol{y}_t = \boldsymbol{\alpha} + \tilde{\mathbf{B}}_t \tilde{\boldsymbol{\varphi}}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_\varepsilon^2 \mathbb{I}_p),$$
$$\tilde{\boldsymbol{\varphi}}_t = \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{\varphi}}_{t-1} + \tilde{\boldsymbol{\Theta}} + \tilde{\boldsymbol{\omega}}_t, \quad \tilde{\boldsymbol{\kappa}}_t \overset{iid}{\sim} \mathcal{N}(0, \tilde{\Upsilon})$$

where $\tilde{\boldsymbol{\varphi}}_t = (\boldsymbol{\varphi}_t, \boldsymbol{\varrho}_t)$ is a $(p + pk + 1) \times 1$ latent process vector and

$$\tilde{\boldsymbol{\Theta}} = \begin{pmatrix} \boldsymbol{\Theta}_{(p+1)\times 1} \\ \boldsymbol{\Psi}_{pk\times 1} \end{pmatrix}_{(p+pk+1)\times 1}$$

is a vector of drift parameters.

We assume independence of error terms in latent variables to give a covariance matrix for the state error $\tilde{\boldsymbol{\omega}}_t$:

$$\tilde{\Upsilon} = \left(\begin{array}{c|c} \Upsilon_{(p+1)\times(p+1)} & 0 \\ \hline 0 & \sigma_\varrho^2 \mathbb{I}_{pk} \end{array}\right)_{(p+pk+1)\times(p+pk+1)}$$

State-Space Hybrid Factor Models for Mortality

Define the following two objects: $\tilde{\mathbf{F}}_t = \bigoplus_{j=1}^{k} [\mathbf{F}_t]_{j,\cdot}$ and $\tilde{\mathbf{f}}_t = vec\left(\mathbf{F}_t^T\right)$ giving:

$$\tilde{\mathbf{F}}_t = \begin{pmatrix} [\mathbf{F}_t]_{1,\cdot} & 0 & 0 & \cdots & 0 \\ 0 & [\mathbf{F}_t]_{2,\cdot} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & [\mathbf{F}_t]_{p,\cdot} \end{pmatrix}_{p \times pk} \quad \text{and } \tilde{\mathbf{f}}_t = \begin{pmatrix} [\mathbf{F}_t]_{1,1} \\ [\mathbf{F}_t]_{1,2} \\ \vdots \\ [\mathbf{F}_t]_{p,k} \end{pmatrix}_{pk \times 1}$$

where $[\mathbf{F}_t]_{j,\cdot}$ and $[\mathbf{F}_t]_{j,m}$ represent the jth row and the element corresponding to jth row and mth column of $\mathbf{F}_t$, respectively.

Consider three cases of model:

- **Case 1:** *Factors in Observation Equation Only*;
- **Case 2:** *Factors in Period Effect State Equation Only*;
- **Case 3:** *Factors in Cohort Effect State Equation Only*.

State-Space Hybrid Factor Models for Mortality

We can now define the different sub-models:

$$\tilde{\mathbf{B}}_{t\ p\times(p+pk+1)} = \begin{cases} \left(\begin{array}{c|c} \mathbf{B}_{p\times(p+1)} & \tilde{\mathbf{F}}_t \end{array}\right) & \text{for } \textbf{Case 1}, \\ \left(\begin{array}{c|c} \mathbf{B}_{p\times(p+1)} & \mathbf{0}_{p\times pk} \end{array}\right) & \text{otherwise,} \end{cases}$$

$$\tilde{\Lambda}_{(p+pk+1)\times(p+pk+1)} = \begin{cases} \left(\begin{array}{c|c} \Lambda_{(p+1)\times(p+1)} & \mathbf{0}_{(p+1)\times pk} \\ \hline \mathbf{0}_{pk\times(p+1)} & \Omega_{pk\times pk} \end{array}\right) & \text{for } \textbf{Case 1}, \\ \left(\begin{array}{c|c} \Lambda_{(p+1)\times(p+1)} & \begin{array}{c} \tilde{\mathbf{f}}_t^T \\ \mathbf{0}_{p\times pk} \end{array} \\ \hline \mathbf{0}_{pk\times(p+1)} & \Omega_{pk\times pk} \end{array}\right) & \text{for } \textbf{Case 2}, \\ \left(\begin{array}{c|c} \Lambda_{(p+1)\times(p+1)} & \begin{array}{c} \mathbf{0}_{1\times pk} \\ \tilde{\mathbf{F}}_t \end{array} \\ \hline \mathbf{0}_{pk\times(p+1)} & \Omega_{pk\times pk} \end{array}\right) & \text{for } \textbf{Case 3}. \end{cases}$$

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

Mortality Data and Demographic Data Description

Country	**Life Expectancy** (E0)	**No. Births**	**Death Rate** (mx)	**No. Deaths**
Austria	1947 - 2014	1871 - 2014	1947 - 2014	1947 - 2014
Belarus	1959 - 2014	1959 - 2014	1959 - 2014	1959 - 2014
Belgium	1841 - 2015	1840 - 2015	1841 - 2015	1841 - 2015
Czech Republic	1950 - 2010	1947 - 2014	1950 - 2014	1950 - 2014
Denmark	1835 - 2014	1835 - 2014	1835 - 2014	1835 - 2014
Estonia	1959 - 2013	1959 - 2013	1959 - 2013	1959 - 2013
Finland	1878 - 2012	1865 - 2012	1878 - 2012	1878 - 2012
France	1816 - 2014	1806 - 2014	1816 - 2014	1816 - 2014
East Germany	1956 - 2013	1946 - 2013	1956 - 2013	1956 - 2013
West Germany	1956 - 2013	1946 - 2013	1956 - 2013	1956 - 2013
Greece	1981 - 2013	1981 - 2013	1981 - 2013	1981 - 2013
Estonia	1959 - 2013	1959 - 2013	1959 - 2013	1959 - 2013
Hungary	1950 - 2014	1950 - 2014	1950 - 2014	1950 - 2014
Iceland	1838 - 2013	1838 - 2013	1838 - 2013	1838 - 2013
Ireland	1950 - 2014	1950 - 2014	1950 - 2014	1950 - 2014

Table: Demographic data available per country (HM Database).

Mortality Data and Demographic Data Description

Country	Life Expectancy (E0)	No. Births	Death Rate (mx)	No. Deaths
Italy	1872 - 2012	1862 - 2012	1872 - 2012	1872 - 2012
Latvia	1959 - 2013	1959 - 2013	1959 - 2013	1959 - 2013
Lithuania	1959 - 2013	1959 - 2013	1959 - 2013	1959 - 2013
Luxembourg	1960 - 2014	1950 - 2014	1960 - 2014	1960 - 2014
Netherlands	1850 - 2012	1850 - 2012	1850 - 2012	1850 - 2012
Norway	1846 - 2014	1846 - 2014	1846 - 2014	1846 - 2014
Poland	1958 - 2014	1958 - 2014	1958 - 2014	1958 - 2014
Portugal	1940 - 2012	1886 - 2012	1940 - 2012	1940 - 2012
Russia	1959 - 2014	1959 - 2014	1959 - 2014	1959 - 2014
Slovakia	1950 - 2014	1950 - 2014	1950 - 2014	1950 - 2014
Slovenia	1983 - 2014	1983 - 2014	1983 - 2014	1983 - 2014
Spain	1908 - 2014	1908 - 2014	1908 - 2014	1908 - 2014
Sweden	1751 - 2014	1747 - 2014	1751 - 2014	1751 - 2014
Switzerland	1876 - 2014	1871 - 2014	1876 - 2014	1876 - 2014
United Kingdom	1922 - 2013	1922 - 2013	1922 - 2013	1922 - 2013
Ukraine	1959 - 2013	1946 - 2013	1959 - 2013	1959 - 2013

Table: Demographic data available per country (HH Database).

Table of contents

Context of Mortality Modelling and Significance

Stochastic Mortality Models Background

State Space Stochastic Mortality Models

Stochastic Mortality Modelling: State-Space Hybrid Factor Models

Mortality Modelling and Demographic Data

State-Space Hybrid Factor Model Results

Results and Analysis

The model estimation was performed by Forward-Backward Kalman Filter within Rao-Blackwellised Adaptive Gibbs Sampler (MCMC).

The models we considered in our studies were of type:

1. [LCC:] Lee-Carter model with the stochastic cohort effect.
2. [DFM-PC:] demographic factor model version of Lee-Carter (Period-Cohort).

The factors are obtained by performing **Probabilistic Principle Component Analaysis PPCA** jointly on the set of data for all countries listed, excluding:
United Kingdom (response variable)
Greece and Slovakia (due to short time series).

Results and Analysis

[DFM-PC:] demographic factor model version of LCC sub-models:

- [DFM-PC-B:] the mean of first principal component of Birth counts as a static parameter, age specific element of ϱ_t;
- [DFM-PC-D-r/s:] the first principal component of Death counts (which is age and country specific) as an exogenous factor, one element of ϱ_t corresponds to a country specific subvector of the component, robust standardisation (s $=$ non-robust standardisation);
- [DFM-PC-Mx-r/s:] the first principal component of Death Rates (which is age and country specific) as an exogenous factor, one element of ϱ_t corresponds to a country specific subvector of the component, robust standardisation (s $=$ non-robust standardisation);

Results and Analysis

Out-of-Sample Forecast Age Specific Log-Death Rates: Performance Analysis

- We choose for the out-of-sample study the last 10 years of the available sample for British Female death rates.
- Model calibration period is $1922 - 2002$
  $\Rightarrow$ forecast performance analysis for $2003 - 2013$

Model	MSE	DIC	$MSEP_{MCMC}$	$MSEP_{Kalman}$
LCC	0.0097	-3627	0.1778	0.1774
DFM-PC-B	**0.0072**	**-6500**	**0.0057**	**0.0062**
DFM-PC-D-r	0.0182	-6380	0.0177	0.0251
DFM-PC-D-s	**0.0065**	**-5996**	**0.0185**	**0.0156**
DFM-PC-Mx-r	**0.0081**	**-8225**	**0.0111**	**0.0129**
DFM-PC-Mx-s	0.0174	-3951	0.0692	0.0285

- **The results confirm that adding demographic features, as additional explanatory variables to the LCC model, improves both in-sample fit out-of-sample fit and therefore the predictability of log death rates.**

Results and Analysis

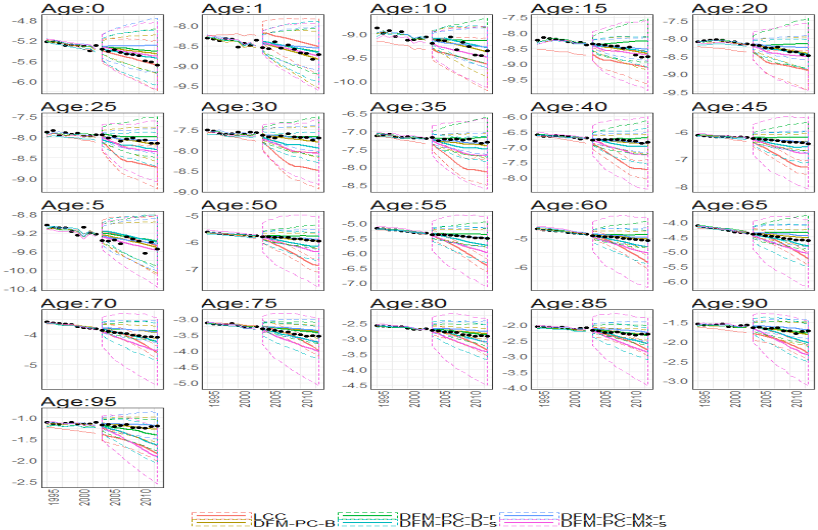

Figure: 10-year out-of-sample forecasted log death (y axis) rates by age with corresponding prediction intervals.

Conclusions

- We explored how to construct a state space formulation of the stochastic mortality models for Period and Cohort factors
- We explored how to extend to Hybrid Multi-Factor Stochastic State-Space Mortality models with Period-Cohort factors as well as demographic regressors.
- We briefly learnt about feature/covariate extraction methods to extract the demographic factors used in the extended HMF Stochastic State-Space Mortality models.
- We see that the standard Lee-Carter stochastic mortality models consistenly under performs in-sampel and out-of-sample in a range of estimation criteria, compared to the new proposed models.
- Lee-Carter Period-Cohort model consistently under estimates log-death rates
- Extended models proposed improve significantly the forecast performance of log-death rates.

Recent Papers on Stochastic Mortality Modelling

1 Fung M.C., Peters G.W., Shevchenko P.V.
A unified approach to mortality modelling using state-space framework: characterisation, identification, estimation and forecasting.
Annals of Actuarial Science. 2017 May:1-47.
Available at SSRN: https://ssrn.com/abstract=2786559

2 Fung M.C. and Peters G.W. and Shevchenko P.V.
A State-Space Estimation of the Lee-Carter Mortality Model and Implications for Annuity Pricing
MODSIM Modelling and Simulation Society. 2015, July.
Available at SSRN: https://ssrn.com/abstract=2699624

3 Toczydlowska D., Peters G.W., Fung M.C. and Shevchenko P.V.
Stochastic Period and Cohort Effect State-Space Mortality Models Incorporating Demographic Factors via Probabilistic Robust Principle Components Risks: Special Issue on "Aging Population Risks".
Available at SSRN: https://ssrn.com/abstract=2977306

4 Fung M.C., Peters G.W. and Shevchenko P.V.
Cohort Effects in Mortality Modelling: A Bayesian State-Space Approach
(March 24, 2017).
Available at SSRN: https://ssrn.com/abstract=2907868