Valuation of Large Variable Annuity Portfolios: Challenges and Potential Solutions

Guojun Gan, PhD, ASA

Department of Mathematics, University of Connecticut, Storrs, CT, USA

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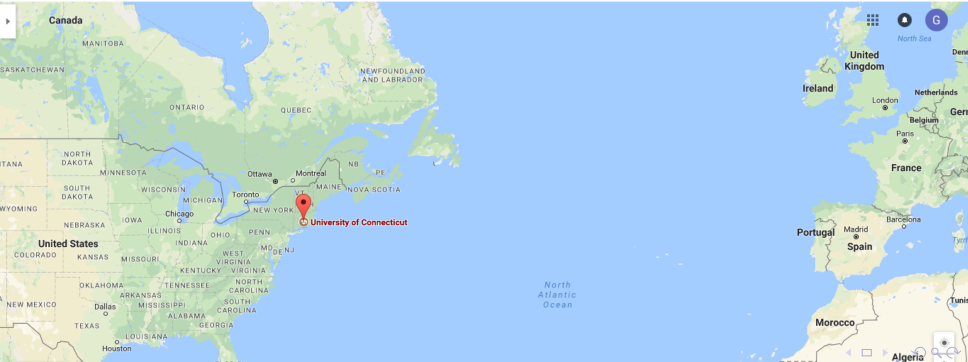
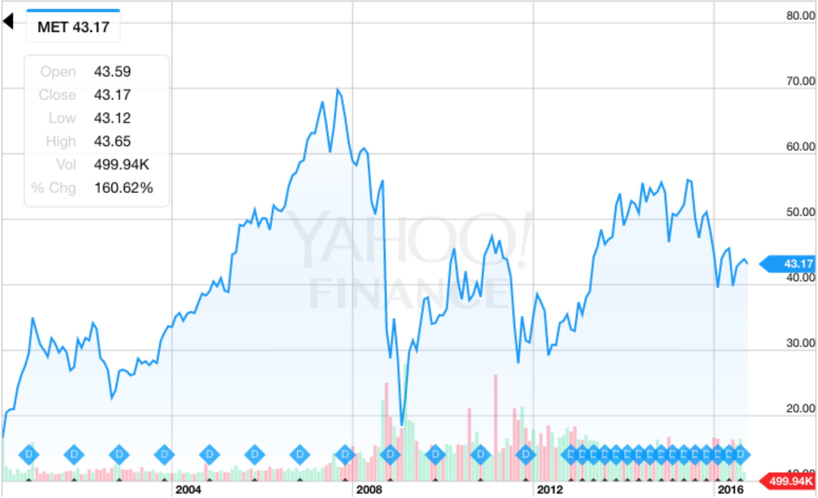


Outline

- Computational problems from variable annuities (VA)
- Metamodeling approaches
- Some numerical examples

MetLife



MET 43.17
Open 43.59
Close 43.17
Low 43.12
High 43.65
Vol 499.94K
% Chg 160.62%
80.00
70.00
60.00
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43.17
40.00
30.00
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499.94K
2004
2008
2012
2016

Prudential

PRU 77.78
Open 78.40
Close 77.78
Low 77.69
High 78.42
Vol 206.36K
% Chg 134.35%
110.00
100.00
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2004
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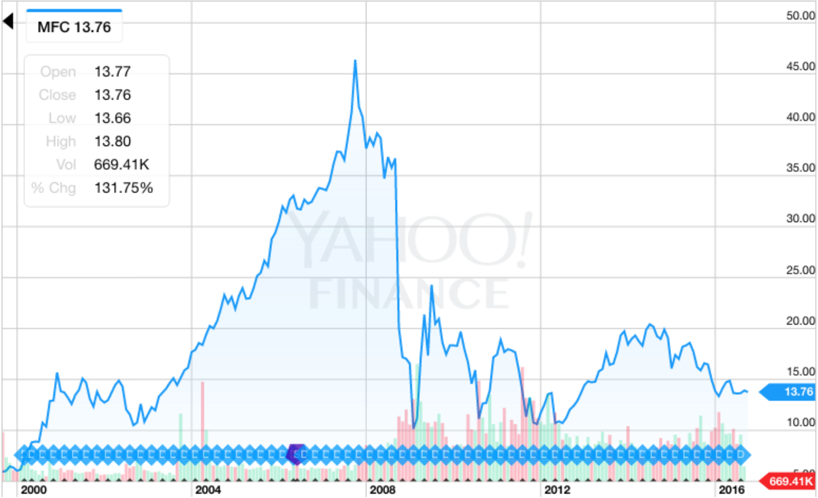
The Hartford

Manulife Financial



Financial risks associated with variable annuities

- Traditional actuarial approaches cannot address these financial risks adequately (Hardy, 2000)
- Dynamic hedging is a popular approach to mitigate these financial risks in practice

Dynamic hedging requires calculating the Greeks of the portfolio

- Due to the complexity of the guarantees, there are in general no closed-form formulas for calculating the Greeks
- Insurance companies resort to Monte Carlo simulation

Monte Carlo simulation is computationally demanding for large portfolios of variable annuities

- Each policy needs to be projected with its own characteristics
- Long-term projection (e.g., 30 years) is involved
- Many economic scenarios (e.g., 1000 scenarios) are required.

(Dardis, 2016)

An example

- 100,000 policies
- 1,000 risk-neutral scenarios
- 360 time steps (monthly steps for 30 years)

The total number of cash flow projections for this portfolio is:

$$1,000 \times 12 \times 30 \times 100,000 = 3.6 \times 10^{10}.$$

Suppose that a computer can process 200,000 cash flow projections per second. Then it would take this computer

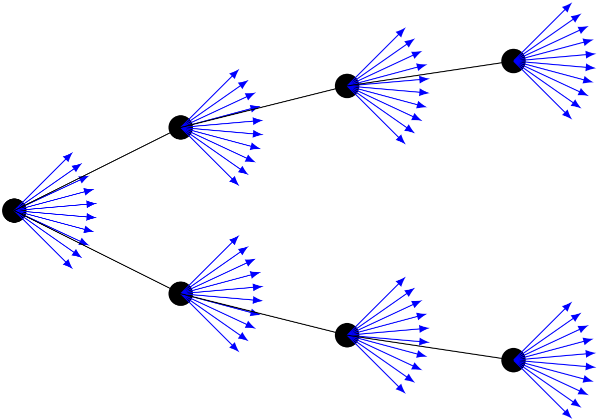
$$\frac{3.6 \times 10^{10} \text{ projections}}{200,000 \text{ projections/second}} = 50 \text{ hours}$$

to process all the cash flows of the portfolio.

Two computational problems related to dynamic hedging

- Calculate partial dollar deltas, partial dollar Rho, and dollar vegas for daily hedging purpose
  - Need to know the Greeks of the liability portfolio within a short time interval (e.g., a second) in order to rebalance the hedge position in a timely manner
- Reflect the dynamic hedging program in quarterly financial reporting
  - A stochastic-on-stochastic framework is usually used to simulate the dynamic hedging program

Nested simulation



An example of nested simulation

- 100,000 policies
- 1,000 inner loop risk neutral scenarios
- 1,000 outer loop real world scenarios
- 360 monthly time steps

$$\boxed{100,000 \times 1,000 \times 1,000 \times 360 \times 360/2 = 6.48 \times 10^{15}!}$$

$$\boxed{\frac{6.48 \times 10^{15} \text{ projections}}{200,000 \text{ projections/second}} \approx 1027 \text{ years!}}$$

(Reynolds and Man, 2008)

Metamodeling Approaches

A metamodel is a model of the Monte Carlo simulation model (Friedman, 2013) that can be used to replace the Monte Carlo simulation model to value the VA contracts in a large portfolio. A metamodeling approach involves four major steps (Barton, 2015):

1. select a small number of representative VA contracts from the portfolio;
2. run the Monte Carlo simulation model to calculate the fair market values of the selected VA contracts;
3. build a metamodel based on the selected VA contracts and the corresponding fair market values; and
4. use the metamodel to estimate the fair market values of all VA contracts in the portfolio.

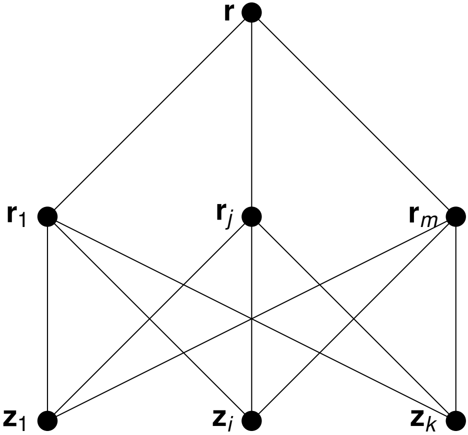
Experimental Design Methods

- Random sampling
- Data clustering
- Latin hypercube sampling
- Conditional Latin hypercube sampling

Metamodel Forms

- Kriging (Gan, 2013; Gan and Lin, 2015; Gan, 2015; Gan and Lin, 2016)
- GB2 regression models (Gan and Valdez, 2017)
- Neural networks (Hejazi and Jackson, 2016)

A two-level metamodeling approach was proposed to address the first computational problem



(Gan and Lin, 2016)

The level-one metamodel I

- Conditional Latin hypercube sampling
- Universal kriging

Under the universal kriging method, the partial dollar Delta of policy $\mathbf{x}_i$ on the hth tradable index when the market levle on the next day is $\mathbf{r}_l$ can be expressed as:

$$\hat{f}(\mathbf{x}_i, \mathbf{r}_l, h) = \boldsymbol{\lambda}_i^T f(Z, \mathbf{r}_l, h) = \sum_{j=1}^{k} \lambda_{ij} f(\mathbf{z}_j, \mathbf{r}_l, h), \quad (1)$$

where $\boldsymbol{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik})^T$ is a vector of kriging weights,

$$\begin{pmatrix} \mathbf{A}(Z) & \mathbf{B}(Z) \\ \mathbf{B}^T(Z) & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\lambda}_i \\ \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}(Z, \mathbf{x}_i) \\ \mathbf{B}(\mathbf{x}_i)^T \end{pmatrix}, \quad (2)$$

The level-one metamodel II

In Equation (2), $\mathbf{A}(Z)$ is a $k \times k$ matrix defined as

$$\mathbf{A}(Z) = \begin{pmatrix} \gamma(\mathbf{z}_1, \mathbf{z}_1) & \gamma(\mathbf{z}_1, \mathbf{z}_2) & \cdots & \gamma(\mathbf{z}_1, \mathbf{z}_k) \\ \gamma(\mathbf{z}_2, \mathbf{z}_1) & \gamma(\mathbf{z}_2, \mathbf{z}_2) & \cdots & \gamma(\mathbf{z}_2, \mathbf{z}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(\mathbf{z}_k, \mathbf{z}_1) & \gamma(\mathbf{z}_k, \mathbf{z}_2) & \cdots & \gamma(\mathbf{z}_k, \mathbf{z}_k) \end{pmatrix}, \tag{3}$$

where $\gamma(\cdot, \cdot)$ is the exponential semivariogram function defined as

$$\gamma(\mathbf{x}, \mathbf{y}) = 1 - \exp\left(-3\frac{\|\mathbf{x} - \mathbf{y}\|}{\beta}\right).$$

The matrix $\mathbf{B}(Z)$ is a $k \times (d+1)$ matrix defined as

$$\mathbf{B}(Z) = \begin{pmatrix} 1 & z_{11} & \cdots & z_{1d} \\ 1 & z_{21} & \cdots & z_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_{k1} & \cdots & z_{kd} \end{pmatrix}. \tag{4}$$

The level-one metamodel III

The column vector $\mathbf{A}(Z, \mathbf{x}_i)$ and the row vector $\mathbf{B}(\mathbf{x}_i)$ are defined as

$$\mathbf{A}(Z, \mathbf{x}_i) = \begin{pmatrix} \gamma(\mathbf{z}_1, \mathbf{x}_i) \\ \gamma(\mathbf{z}_2, \mathbf{x}_i) \\ \vdots \\ \gamma(\mathbf{z}_k, \mathbf{x}_i) \end{pmatrix} \tag{5}$$

and

$$\mathbf{B}(\mathbf{x}_i) = \begin{pmatrix} 1 & x_{i1} & \cdots & x_{id} \end{pmatrix}, \tag{6}$$

respectively.

The level-two metamodel I

- Latin hypercube sampling
- Ordinary kriging

Under the ordinary kriging method, the partial dollar Delta of the portfolio on the hth tradable index when the market level is $\mathbf{r}$ can be expressed as

$$\hat{g}(\mathbf{r}, h) = \sum_{l=1}^{m} w_l \cdot \hat{f}(X, \mathbf{r}_l, h), \tag{7}$$

where $\hat{f}(X, \mathbf{r}_l, h)$ is the partial dollar Delta of the portfolio on the hth tradable index when the market level is $\mathbf{r}_l$ and $w_1, w_2, \ldots, w_m$ are the kriging weights.

$$\begin{pmatrix} V_{11} & \cdots & V_{1m} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{m1} & \cdots & V_{mm} & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_m \\ \theta \end{pmatrix} = \begin{pmatrix} D_1 \\ \vdots \\ D_m \\ 1 \end{pmatrix}, \tag{8}$$

The level-two metamodel II

where θ is the Lagrange multiplier to ensure the sum of the kriging weights equal to one,

$$V_{ls} = 1 - \exp\left(-\frac{3}{\beta}\|\mathbf{r}_l - \mathbf{r}_s\|\right), \quad l, s = 1, 2, \ldots, m,$$

and

$$D_l = 1 - \exp\left(-\frac{3}{\beta}\|\mathbf{r} - \mathbf{r}_l\|\right), \quad j = 1, 2, \ldots, m.$$

A portfolio of synthetic variable annuity policies

Attribute	Values
Guarantee type	{DBRP, DBRU, WB, WBSU, MB}
Gender	{Male, Female}
Birth date range	[1/1/1950, 1/1/1980]
Issue date range	1/1/2000 1/1/2014]
Valuation date	1/1/2014
Maturity range	[15, 30]
Account value range	[50000, 500000]
Maturity	{10, 11, 12, . . ., 25}
Number of investment funds	10

All investment funds are mapped to five indices

Fund	US Large	US Small	Intl Equity	Fixed Income	Money Market
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1
6	0.6	0.4	0	0	0
7	0.5	0	0.5	0	0
8	0.5	0	0	0.5	0
9	0	0.3	0.7	0	0
10	0.2	0.2	0.2	0.2	0.2

Some validation measures

$$\text{RMSE}(h) = \sqrt{\frac{1}{J}\sum_{l=1}^{J}(\hat{y}_{lh} - y_{lh})^2}, \tag{9}$$

$$\text{RAAE}(h) = \frac{\sum_{l=1}^{J}|\hat{y}_{lh} - y_{lh}|}{J \times \sigma_h}, \tag{10}$$

$$R_h^2 = 1 - \frac{\sum_{l=1}^{J}(\hat{y}_{lh} - y_{lh})^2}{\sum_{l=1}^{J}(\mu_h - y_{lh})^2} \tag{11}$$

$$\text{RMAE}(h) = \frac{\max_{1\le l\le J}|\hat{y}_{lh} - y_{lh}|}{\sigma_h}, \tag{12}$$

$$\text{APE}(h) = \frac{1}{J}\sum_{l=1}^{J}\frac{\hat{y}_{lh} - y_{lh}}{y_{lh}}, \tag{13}$$

$$\text{AAPE}(h) = \frac{1}{J}\sum_{l=1}^{J}\left|\frac{\hat{y}_{lh} - y_{lh}}{y_{lh}}\right|, \tag{14}$$

The metamodels in different levels are validated separately and jointly

To validate the level-one metamodel, we set

$$\hat{y}_{lh} = \hat{f}(X, \mathbf{r}_l, h), \quad y_{lh} = f(X, \mathbf{r}_l, h), \quad J = k,$$

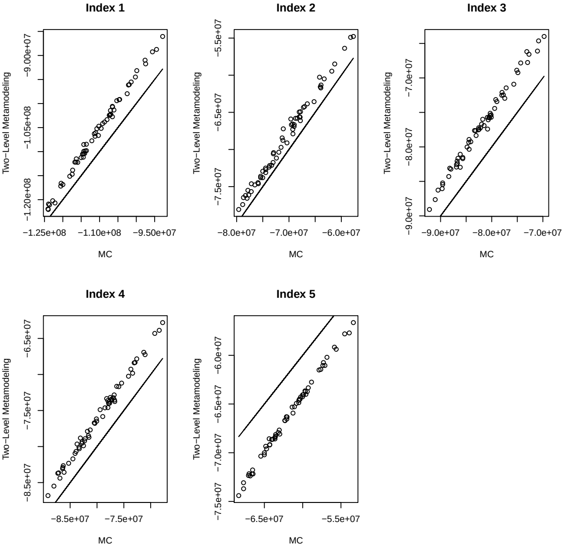
To validate the level-two metamodel alone, we set

$$\hat{y}_{lh} = g(\mathbf{s}_l, h), \quad y_{lh} = f(X, \mathbf{s}_l, h), \quad J = m,$$

To validate the two-level metamodeling approach with the level-one and level-two metamodels together, we set

$$\hat{y}_{lh} = \hat{g}(\mathbf{s}_l, h), \quad y_{lh} = f(X, \mathbf{s}_l, h), \quad J = m,$$

There are some biases when $k = 220$



Accuracy of the two-level metamodeling approach with $k = 220$ and $m = 50$.

	RMSE	RAAE	R-Squared	RMAE	APE	AAPE
Index 1	4,698,664	0.5811	0.6316	0.8675	-0.0419	0.0419
Index 2	2,302,026	0.4021	0.8188	0.7146	-0.0317	0.0317
Index 3	4,988,233	0.9079	0.1395	1.2809	-0.0609	0.0609
Index 4	3,746,782	0.7114	0.4688	0.9593	-0.047	0.047
Index 5	4,496,789	1.2115	-0.5247	1.6475	0.0717	0.0717

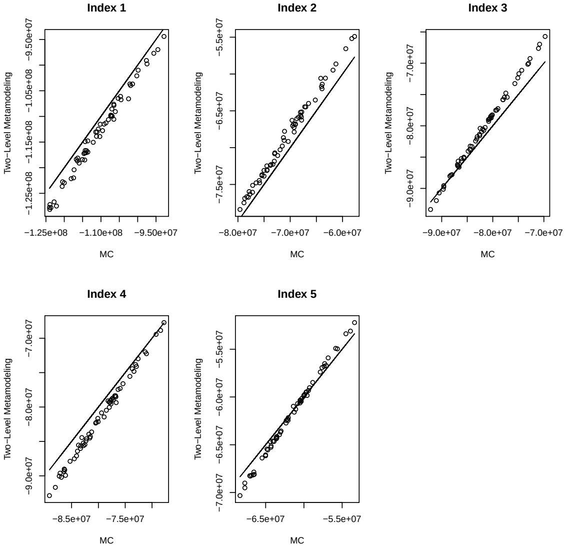
Accuracy of the level-one metamodel with $k = 220$ and $m = 50$.

	RMSE	RAAE	R-Squared	RMAE	APE	AAPE
Index 1	4,901,431	0.3274	0.8703	0.6525	-0.043	0.0431
Index 2	3,085,133	0.2763	0.8884	0.6799	-0.0335	0.0378
Index 3	5,076,421	0.4822	0.7307	0.9234	-0.0602	0.061
Index 4	3,814,150	0.3984	0.8175	0.6768	-0.0466	0.0466
Index 5	4,579,153	0.7101	0.4679	1.0208	0.0729	0.0729

Accuracy of the level-two metamodel with $k = 220$ and $m = 50$.

	RMSE	RAAE	R-Squared	RMAE	APE	AAPE
Index 1	632,799	0.066	0.9933	0.1923	-0.0012	0.0047
Index 2	428,126	0.0655	0.9937	0.1836	-0.0011	0.0051
Index 3	364,905	0.0515	0.9954	0.1664	-0.0015	0.0035
Index 4	390,177	0.06	0.9942	0.1712	-0.0014	0.004
Index 5	311,483	0.0694	0.9927	0.2025	-9.00E-04	0.0042

Biases reduced when $k = 440$



The accuracy of the two-level metamodeling approach when $k = 440$ and $m = 50$.

	RMSE	RAAE	R-Squared	RMAE	APE	AAPE
Index 1	2,781,176	0.3327	0.8709	0.6853	0.0233	0.0233
Index 2	2,124,997	0.3718	0.8456	0.6566	-0.0293	0.0293
Index 3	1,808,466	0.2776	0.8869	0.7427	-0.0182	0.0193
Index 4	2,142,709	0.3766	0.8263	0.7322	0.024	0.024
Index 5	803,153	0.1776	0.9514	0.5359	0.004	0.0104

Accuracy of the level-one metamodel when $k = 440$ and $m = 50$.

	RMSE	RAAE	R-Squared	RMAE	APE	AAPE
Index 1	3,733,350	0.226	0.9247	0.6062	0.0246	0.0283
Index 2	2,530,037	0.2294	0.925	0.5583	-0.0302	0.0316
Index 3	1,953,643	0.164	0.9601	0.3728	-0.0195	0.0218
Index 4	2,479,114	0.231	0.9229	0.6286	0.0246	0.0255
Index 5	803,141	0.1044	0.9836	0.3303	0.0036	0.0105

Accuracy of the level-two metamodel when $k = 440$ and $m = 50$.

	RMSE	RAAE	R-Squared	RMAE	APE	AAPE
Index 1	632,799	0.066	0.9933	0.1923	-0.0012	0.0047
Index 2	428,126	0.0655	0.9937	0.1836	-0.0011	0.0051
Index 3	364,905	0.0515	0.9954	0.1664	-0.0015	0.0035
Index 4	390,177	0.06	0.9942	0.1712	-0.0014	0.004
Index 5	311,483	0.0694	0.9927	0.2025	-9.00E-04	0.0042

Runtime of the two-level metamodeling approach

	$k = 220$, $m = 50$	$k = 440$, $m = 50$	Full, $m = 60$
`clhs`	138.75	169.44	NA
Monte Carlo	639.55	1,279.10	42,090.08
Level-One Metamodel	8.12	16.66	NA
`lhs`	0.02	0.02	NA
Level-Two Metamodel	0.19	0.33	NA
Total (seconds)	786.63	1,465.55	42,090.08
Total (hours)	0.22	0.41	11.69

Future work

- How to determine the sample size?
- Error estimation.
- Efficient experimental design methods.
- Many partial dollar Deltas are zero.
- Other applications (e.g., economic capital)

Please get in touch

Guojun Gan, PhD, ASA

Department of Mathematics

University of Connecticut

Storrs, CT

guojun.gan@uconn.edu

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